

Selection intensity and risk-dominant strategy: A two-strategy stochastic evolutionary game dynamics in finite population



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ABSTRACT

Stochastic evolutionary game dynamics with weak selection in finite population has been studied and it has been used to explain the emergence of cooperation. In this paper, following the previous studies, the diffusion approximation of a two-strategy stochastic evolutionary game dynamics in finite population that includes a small mutation rate between two strategies is investigated, where we assume that these two strategies are both strict Nash equilibrium (NE). Our main goal is to partially reveal the effect of selection intensity on the stochastic evolutionary game dynamics. Through the analysis of potential function of the stationary distribution, our main result shows that for all possible situations with that the selection intensity is not zero (that includes the strong selection), if a strategy is a risk-dominant NE, then its expected fitness with respect to the stationary distribution must be larger than that of other strategy. This result not only extends the previous results but also provides some useful insights for understanding the significance of selection intensity in stochastic evolutionary game dynamics in finite population.

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1. Introduction

Evolutionary game theory has been successfully applied since its introduction by Maynard Smith and Price [1]. Recently, the stochastic evolutionary game dynamics in the finite population was investigated [2,3], and it has been used to explain the evolutionary emergence of cooperation. Nowak et al. [2] studied a stochastic process for games with two (pure) strategies, denoted by A and B , respectively, when population size is fixed at N . They introduced the concept of one-third law to show why the selection could favor a single mutant using strategy A replacing a resident population using the other pure strategy B . In this basic form, the one-third law connects the Nash equilibrium (NE) concepts of this two-strategy game to stochastic properties of the birth-death Markov system known as the Moran process [2–4]. Specifically, Nowak et al. [2] showed that, when A and B are both strict NE, population size is large and the selection is weak, the probability that this replacement occurs under the Moran process is higher than that obtained from neutral drift if the completely mixed NE of this game plays strategy A less than one-third of the time (see also [3,5]). Taylor et al. [6] investigated the evolutionary significance of one-third law in the Moran process (see also [7]).

The diffusion approximation of the stochastic evolutionary game dynamics based on the Moran process was first studied by Traulsen et al. [8,9]. For the situation with weak selection, their main results exactly match the Nowak et al.'s results [2]. Zheng et al. [10] also used the diffusion approximation of stochastic evolutionary game dynamics to reveal the evolutionary

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significance of the one-third law, and they showed that the one-third law implies that the difference between the average fitness of strategies A and B is positive when a single individual using A is initially present in a large population that consists otherwise of individuals using B . On the other hand, Zheng et al. [10] connected the risk-dominance of strategy A to its larger expected fitness with respect to the stationary distribution of the diffusion approximation that includes a small mutation rate between the two strategies. All of these results are based on the assumption that the selection is weak. However, Traulsen et al. considered stochastic evolutionary game dynamics for individual and group selection at any intensity [11,12]. In this paper, to show the effect of selection intensity on the stochastic evolutionary game dynamics, or the relationship between the risk-dominance, expected fitness with respect to the stationary distribution and selection intensity, we will study how the selection intensity influences the characteristics of potential function of the stationary distribution. Our results may provide some useful insights into understanding the significance of selection intensity in stochastic evolutionary game dynamics in finite population.

We have also noticed that the evolutionary graph theory and spatial games have been developed to reveal the evolution of cooperation in structured population [3]. Some studies on the stochastic game dynamics in structured population suggested that the selection mechanism for promoting cooperation in structured population could be significantly different from that of unstructured population [13–18], and some researchers also considered the effect of selection intensity on the evolutionary graph theory [19]. Although in this study we consider only the situation with well-mixed population, we will extend our analysis to the situation with structured population in the next step.

2. Model and analysis

Similar to Traulsen and Hauert [7] (see also [2,5,10]), we consider also a simple two-strategy stochastic evolutionary game dynamics in a finite population, where the payoff matrix is given by

$$\begin{array}{c} A \\ B \end{array} \begin{array}{cc} A & B \\ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \end{array}, \quad (1)$$

where the payoff of an individual displaying strategy A is a (b) when it plays against an individual displaying strategy A (B) and the payoff of an individual displaying strategy B is c (d) when it plays against an individual displaying strategy A (B). Without loss of generality, we here assume that all the entries of the payoff matrix are positive. The population size, N , is assumed to be fixed, the number of individuals displaying is denoted by j and the number of individuals displaying B is $N-j$. For the frequency-dependent Moran process, the expected payoffs of A and B in random pairwise interactions, denoted by f_j and g_j , respectively, are

$$\begin{aligned} f_j &= (1 - \omega) + \omega \frac{a(j-1) + b(N-j)}{N-1}, \\ g_j &= (1 - \omega) + \omega \frac{cj + d(N-j-1)}{N-1}, \end{aligned} \quad (2)$$

where the parameter ω denotes the selection intensity with $0 \leq \omega \leq 1$, which represents the contribution of the game to individual fitness [2,3].

We here still assume that, at each time step, an individual is chosen for reproduction proportional to its fitness, and that the one identical offspring produced replaces another randomly chosen individual [2]. For this system, selection is said to favor A replacing B if, when there is initially one A individual in the population (or the initial frequency of A is $1/N$), the fixation probability of A , denoted by ρ_A , is larger than $1/N$, which is the fixation probability under neutral selection ($\omega=0$). Nowak et al. [2] have shown that, under the Moran process with weak selection (specifically, $N\omega \ll 1$), selection favors A replacing B if and only if $(N-2)a + (2N-1)b > (N+1)c + (2N-4)d$, and that if both A and B are strict NE for the payoff matrix given in Eq. (1) (this means that the payoff matrix satisfies $a > c$ and $d > b$), then $\rho_A > 1/N$ if $x^* < 1/3$, where $x^* = (b-d)/(b-d+c-a)$ is the frequency of A at the completely mixed NE of this game [2]. This is also called the one-third law [3]. Nowak et al. [2] also pointed out that, when both A and B are strict NE, $\rho_A > \rho_B$ (where ρ_B denotes the fixation probability of strategy B , the inequality $\rho_A > \rho_B$ means A is more likely to replace B than vice versa) under the Moran process with weak selection and large population size if and only if A is risk dominant (i.e. $x^* < 1/2$). In this study, we still assume both A and B are strict NE.

For large N , the fitness of A and B can be approximated as

$$\begin{aligned} f(x) &\approx (1 - \omega) + \omega(xa + (1-x)b), \\ g(x) &\approx (1 - \omega) + \omega(xc + (1-x)d), \end{aligned} \quad (3)$$

respectively, where x denotes the frequency of A , $x = j/N$. Thus, it is easy to see that for the diffusion approximation of this process, the transition probabilities from x to $x + 1/N$ and from x to $x - 1/N$ can be given by

$$\pi^+(x) = (1-x) \frac{xf(x)(1-\mu) + (1-x)g(x)\mu}{xf(x) + (1-x)g(x)},$$

$$\pi^-(x) = x \frac{xf(x)\mu + (1-x)g(x)(1-\mu)}{xf(x) + (1-x)g(x)}, \tag{4}$$

where the parameter μ denotes the mutation rate from A to B as well as from B to A [7–10]. Let $\phi(x; t)$ denote the probability density distribution that the frequency of strategy A equals exactly x at time t . Then, we have

$$\begin{aligned} \phi(x; t + 1/N) - \phi(x; t) &= \phi(x - 1/N; t)\pi^+(x - 1/N) \\ &\quad + \phi(x + 1/N; t)\pi^-(x + 1/N) \\ &\quad - \phi(x; t)\pi^-(x) - \phi(x; t)\pi^+(x). \end{aligned} \tag{5}$$

This implies that the Fokker–Planck approximation of $\phi(x; t)$ can be given by

$$\frac{\partial \phi(x; t)}{\partial t} = -\frac{\partial}{\partial x} D^{(1)}(x)\phi(x; t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x)\phi(x; t), \tag{6}$$

where

$$\begin{aligned} D^{(1)}(x) &= \pi^+(x) - \pi^-(x) \\ &= \frac{x(1-x)(f(x) - g(x)) + \mu(-xf(x) + (1-x)g(x))}{xf(x) + (1-x)g(x)}, \\ D^{(2)}(x) &= \frac{\pi^+(x) + \pi^-(x)}{2N} \\ &= \frac{1}{2N} \frac{x(1-x)(f(x) + g(x)) + (2x-1)\mu(xf(x) - (1-x)g(x))}{xf(x) + (1-x)g(x)} \end{aligned} \tag{7}$$

(the mathematical derivations of Eqs. (6) and (7) are shown in the Appendix). It is easy to see that the stationary solution of Fokker–Planck equation Eq. (6), which is defined as $\lim_{t \rightarrow \infty} \phi(x; t) = \phi(x)$, can be expressed as

$$\phi(x) = \frac{C}{D^{(2)}(x)} \exp\left(\int_0^x \frac{D^{(1)}(s)}{D^{(2)}(s)} ds\right), \tag{8}$$

where $C = [\int_0^1 (1/D^{(2)}(x)) \exp(\int^x (D^{(1)}(s)/D^{(2)}(s)) ds)]^{-1}$ is the normalization constant [20]. The potential function of the stationary distribution $\phi(x)$, denoted by $U(x)$, is

$$U(x) = \ln D^{(2)}(x) - \int^x \frac{D^{(1)}(s)}{D^{(2)}(s)} ds. \tag{9}$$

[20].

Under the Fokker–Planck equation Eq. (6), the expected frequency of strategy A with respect to the stationary distribution $\phi(x)$, denoted by $\langle x \rangle$, is $\langle x \rangle = \int_0^1 x\phi(x)dx$, and expected difference between the fitness of strategies A and B under the stationary distribution $\phi(x)$ is $\langle f(x) - g(x) \rangle = \omega\alpha(\langle x \rangle - x^*)$, where $\alpha = a - b - c + d > 0$ (since we assume both A and B are strict NE). For the situation with weak selection, Zheng et al. [10] have shown that strategy A has a higher expected fitness than B if and only if it is the risk-dominant NE (or $\langle x \rangle > 1/2$ if $x^* < 1/2$). We here will show how the expected frequency $\langle x \rangle$ is influenced by the change of selection intensity ω through the analysis of potential $U(x)$.

Case 1. Neutral selection ($\omega=0$)

For $\omega=0$, we have

$$\int^s \frac{D^{(1)}(s)}{D^{(2)}(s)} ds = \frac{N\mu}{1-2\mu} \ln(2x(1-x) + (1-2x)^2\mu).$$

Thus, $U(x)$ can be expressed as

$$U(x) = \ln \frac{2x(1-x) + \mu(1-2x)^2}{2N} - \frac{N\mu}{1-2\mu} \ln(2x(1-x) + (1-2x)^2\mu).$$

Notice that the solution of equation $dU(x)/dx = 0$, denoted by \hat{x} , is $\hat{x} = 1/2$ and it corresponds to a maximum (or minimum) of $U(x)$ if $\mu < 1/(N+2)$ (or $\mu > 1/(N+2)$) (Fig. 1 in which we let $N=98$), and that $U(x)$ is strictly symmetric about $1/2$. Thus, we must have that the expectation of x under the stationary distribution $\phi(x)$ is $\langle x \rangle = 1/2$ (this means that $\phi(x)$ is symmetric about $x = 1/2$, then $\langle x \rangle = \int_0^1 x\phi(x)dx = \int_0^{1/2} x\phi(x)dx + \int_{1/2}^1 (1-x)\phi(x)dx = 1/2$).

Case 2. Weak selection ($N\omega \ll 1$)

For the weak selection with $N\omega \ll 1$ but $\omega \neq 0$, the functions $D^{(1)}(x)$ and $D^{(2)}(x)$ can be approximated as

$$\begin{aligned} D^{(1)}(x) &\approx x(1-x)\omega\alpha(x-x^*) + (1-2x)\mu, \\ D^{(2)}(x) &\approx \frac{1}{2N} [2x(1-x) + (1-2x)^2\mu], \end{aligned}$$

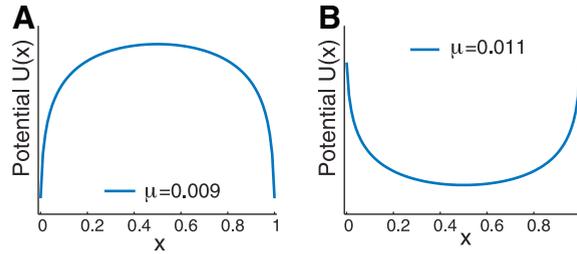


Fig. 1. The effect of mutation rate on the potential $U(x)$ under the neutral selection. For the situation with $\omega=0$, the characteristics of potential $U(x)$ sensitively depends on the size of mutation rate. As an example, we here take the population size $N=98$. The potential $U(x)$ has the maximum at $x = 1/2$ when $\mu = 0.009 < 1/100$ (panel A), and the minimum at $x = 1/2$ when $\mu = 0.011 > 1/100$ (panel B).

respectively, where $\alpha = a - b - c + d > 0$ (since we assume that strategies A and B are both strict NE). So, the potential $U(x)$ can be expressed as

$$U(x) = \ln \frac{2x(1-x) + (1-2x)^2\mu}{2N} - 2N \int^x \frac{s(1-s)\omega\alpha(s-x^*) + (1-2s)\mu}{2s(1-s) + (1-2s)^2\mu} ds.$$

Notice that

$$\begin{aligned} \int^x \frac{s(1-s)\omega\alpha(s-x^*) + (1-2s)\mu}{2s(1-s) + (1-2s)^2\mu} ds &= \int^x \frac{s(1-s)(s-1/2)\omega\alpha + (1-2s)\mu}{2s(1-s) + (1-2s)^2\mu} ds \\ &\quad + \omega\alpha\left(\frac{1}{2} - x^*\right) \int^x \frac{s(1-s)}{2s(1-s) + (1-2s)^2\mu} ds, \end{aligned}$$

where the term

$$2N \int^x \frac{s(1-s)(s-1/2)\omega\alpha + (1-2s)\mu}{2s(1-s) + (1-2s)^2\mu} ds$$

is symmetric about $x = 1/2$ and the term

$$\omega\alpha\left(\frac{1}{2} - x^*\right) \int^x \frac{s(1-s)}{2s(1-s) + (1-2s)^2\mu} ds$$

is positive function if $0 < x^* < 1/2$ that increases in x . Thus, for $0 < x^* < 1/2$, there must exist a \hat{x} in the interval $0 < \hat{x} < 1/2$ such that the potential $U(x)$ has the maximum at $x = \hat{x}$ (where we take that the mutation rate satisfies $\mu < 1/(N+2)$). It is also easy to see that $\hat{x} = 1/2$ if $x^* = 1/2$. This strongly suggests that for $x^* < 1/2$, we have $\langle x \rangle = C \int_0^1 x e^{-U(x)} dx > 1/2$ (this implies $\int_0^{1/2} \phi(x) dx < \int_{1/2}^1 \phi(x) dx$ if $x^* < 1/2$). Similarly, we have $\langle x \rangle < 1/2$ if $x^* > 1/2$. Based on these analyses, when the selection is weak, $\langle f(x) \rangle > \langle g(x) \rangle$ if $x^* < 1/2$ (or $\langle f(x) \rangle < \langle g(x) \rangle$ if $x^* > 1/2$) [10].

Case 3. Effect of selection intensity on the potential $U(x)$ in general situation

To show the effect of ω on potential $U(x)$, we here assume the mutation rate μ is small enough. Based on this assumption, $D^{(1)}(x)$ and $D^{(2)}(x)$ can be approximated as

$$\begin{aligned} D^{(1)}(x) &\approx \frac{x(1-x)(f(x) - g(x))}{xf(x) + (1-x)g(x)}, \\ D^{(2)}(x) &\approx \frac{1}{2N} \frac{x(1-x)(f(x) + g(x))}{xf(x) + (1-x)g(x)}. \end{aligned}$$

So, the potential $U(x)$ is

$$U(x) = \ln \frac{1}{2N} \cdot \frac{x(1-x)(f(x) + g(x))}{xf(x) + (1-x)g(x)} - 2N \int^x \frac{f(s) - g(s)}{f(s) + g(s)} ds, \tag{10}$$

where the terms $f(s) - g(s)$ and $f(s) + g(s)$ can be expressed as

$$\begin{aligned} f(s) - g(s) &= \omega\alpha(s - x^*), \\ f(s) + g(s) &= \omega\beta s + \omega(b + d) + 2(1 - \omega), \end{aligned}$$

where $\beta = a - b + c - d$. Notice that

$$\lim_{x \rightarrow x^*} \frac{d}{dx} \ln \frac{x(1-x)(f(x) + g(x))}{xf(x) + (1-x)g(x)} \begin{cases} < 0 & \text{if } x^* < \frac{1}{2} \\ = 0 & \text{if } x^* = \frac{1}{2} \\ > 0 & \text{if } x^* > \frac{1}{2} \end{cases}.$$

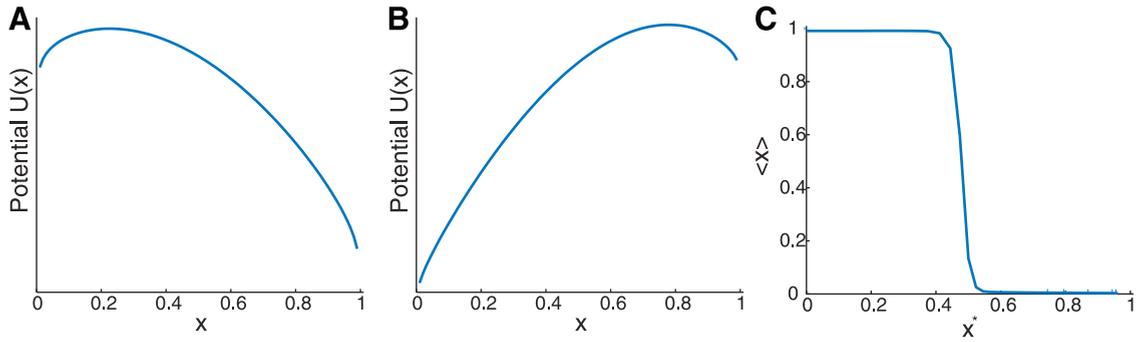


Fig. 2. The relationships between x^* and potential $U(x)$, and between x^* and $\langle x \rangle$ for all possible situations with $\omega > 0$. As an example with payoff matrix $\begin{pmatrix} 2 & 1 \\ 1 & d \end{pmatrix}$ (where $d > 1$), panels A and B show the potential $U(x)$ for $d=1.2$ with $x^* = 1/6$ and for $d=5$ with $x^* = 4/5$, respectively. Panel C shows how the expectation $\langle x \rangle$ changes with x^* .

Thus, for large population size N , the solution of equation $dU(x)/dx = 0$, denoted by \hat{x} , must be close to x^* with $\lim_{N \rightarrow \infty} \hat{x} = x^*$, and $\hat{x} = x^*$ if $x^* = 1/2$, $\hat{x} < x^*$ if $x^* < 1/2$ and $\hat{x} > x^*$ if $x^* > 1/2$. Similarly, we have also that the sign $d^2U(x)/dx^2$ is negative for large N (since $\alpha > 0$). So, the change of selection intensity ω cannot influence the nature of potential $U(x)$. This also strongly implies that, for all possible selection intensity $\omega > 0$, when the mutation rate μ is very small and population size N is large, the expected value of x with respect to the stationary distribution $\phi(x)$, $\langle x \rangle = \int_0^1 x\phi(x)dx$, is larger than $1/2$ if $x^* < 1/2$, or $\langle x \rangle < 1/2$ if $x^* > 1/2$. This also implies that for the situation with small mutation rate and large population size, strategy A has a higher expected fitness with respect to the stationary distribution $\phi(x)$ than B if and only if it is the risk-dominant NE. Obviously, this result is exactly same to the situation with weak selection (see also [10]). So, for the stochastic evolutionary game dynamics in finite population with all possible selection intensity, the risk-dominant NE will have a higher expected fitness.

As a special example, we consider a simple situation with payoff matrix $\begin{pmatrix} 2 & 1 \\ 1 & d \end{pmatrix}$ with $d > 1$. In this example, we have $x^* = (d - 1)/d$. The potential $U(x)$ with $x^* = 1/6$ (where $d=1.2$) and with $x^* = 4/5$ (where $d=5$) are plotted in Fig. 2A and B, respectively. For the relationship between $\langle x \rangle$ and x^* (see Fig. 2C), we can see that not only $\langle x \rangle > 1/2$ (or $\langle x \rangle < 1/2$) if $x^* < 1/2$ (or $x^* > 1/2$) but also the change of $\langle x \rangle$ has obvious critical characteristics when x^* is near to $1/2$. All of these results show clearly that the theoretical analysis is correct.

3. Conclusion

In this study, following Traulsen and Hauert [7] (see also [8,9]) and Zheng et al. [10], the diffusion approximation of a two-strategy (strategies A and B) stochastic evolutionary game dynamics in finite population that includes a small mutation rate between strategies A and B is investigated, where we assume that strategies A and B are both strict NE. For this system, it is easy to obtain its stationary distribution using the Fokker–Planck equation [20]. We mainly focus our attention on how the change of selection intensity influences the characteristics of potential function of the stationary distribution since the nature of the stationary distribution is mainly determined by its potential function. Our main results show that, when the mutation rate is small enough, (i) the potential function is an upper convex function in the interval $0 < x < 1$ (where x denotes the frequency of strategy A) for all possible selection intensity, and it has the maximum at $x = \hat{x}$; (ii) if the selection is neutral, the potential function is exactly symmetric about $x = 1/2$, and $\hat{x} = 1/2$; (iii) if the selection is weak (or the selection intensity is small enough), the potential function is almost symmetric about $x = 1/2$; and (iv) if the selection is not neutral, then \hat{x} must be less (or larger) than $1/2$ if strategy A (or B) is risk-dominant. All of these results strongly imply that for all possible $\omega > 0$, the expectation of x with respect to the stationary distribution $\phi(x)$ is larger (or less) than $1/2$ if $x^* < 1/2$ (or $x^* > 1/2$); or the expected fitness of strategy A with respect to the stationary distribution is larger than that of strategy B if A is risk-dominant, and, conversely, the expected fitness of B is larger than that of A if B is risk-dominant. These results give an alternative meaning to the notation of selection favoring a phenotype more than other originally considered in terms of their fixation probabilities by Nowak et al. [2]. Obviously, our results should be also considered to be an extension of Zheng et al.’s [10] results.

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Appendix

Derivation of Eqs. (6) and (7). Notice that

$$\begin{aligned}\phi(x; t + 1/N) &\approx \phi(x; t) + \frac{\partial}{\partial t}\phi(x; t)\frac{1}{N} \\ &\Rightarrow \phi(x; t + 1/N) - \phi(x; t) \approx \frac{\partial}{\partial t}\phi(x; t)\frac{1}{N}, \\ \phi(x \pm 1/N; t) &\approx \phi(x; t) \pm \frac{\partial}{\partial x}\phi(x; t)\frac{1}{N} + \frac{\partial^2}{\partial x^2}\phi(x; t)\frac{1}{2N^2}, \\ \pi^\pm(x \mp 1/N) &\approx \pi^\pm(x) \mp \frac{\partial}{\partial x}\pi^\pm(x)\frac{1}{N} + \frac{\partial^2}{\partial x^2}\pi^\pm(x)\frac{1}{2N^2},\end{aligned}$$

and that

$$\begin{aligned}\phi(x - 1/N; t)\pi^+(x - 1/N) &\approx \phi(x; t)\pi^+(x) + \phi(x; t)\left(-\frac{\partial}{\partial x}\pi^+(x)\frac{1}{N} + \frac{\partial^2}{\partial x^2}\pi^+(x)\frac{1}{2N^2}\right) \\ &\quad - \pi^+(x)\frac{\partial}{\partial x}\phi(x; t)\frac{1}{N} + \left(\frac{\partial}{\partial x}\phi(x; t)\right)\left(\frac{\partial}{\partial x}\pi^+(x)\right)\frac{1}{N^2} + \pi^+(x)\frac{\partial^2}{\partial x^2}\phi(x; t)\frac{1}{2N^2}, \\ \phi(x + 1/N; t)\pi^-(x + 1/N) &\approx \phi(x; t)\pi^-(x) + \phi(x; t)\left(\frac{\partial}{\partial x}\pi^-(x)\frac{1}{N} + \frac{\partial^2}{\partial x^2}\pi^-(x)\frac{1}{2N^2}\right) \\ &\quad + \pi^-(x)\frac{\partial}{\partial x}\phi(x; t)\frac{1}{N} + \left(\frac{\partial}{\partial x}\phi(x; t)\right)\left(\frac{\partial}{\partial x}\pi^-(x)\right)\frac{1}{N^2} + \pi^-(x)\frac{\partial^2}{\partial x^2}\phi(x; t)\frac{1}{2N^2}.\end{aligned}$$

Then, the right-hand side of Eq. (5) up to second order terms becomes

$$\begin{aligned}&\phi(x; t)\left(-\frac{\partial}{\partial x}\pi^+(x)\frac{1}{N} + \frac{\partial^2}{\partial x^2}\pi^+(x)\frac{1}{2N^2}\right) - \pi^+(x)\frac{\partial}{\partial x}\phi(x; t)\frac{1}{N} \\ &\quad + \left(\frac{\partial}{\partial x}\phi(x; t)\right)\left(\frac{\partial}{\partial x}\pi^+(x)\right)\frac{1}{N^2} + \pi^+(x)\frac{\partial^2}{\partial x^2}\phi(x; t)\frac{1}{2N^2} \\ &\quad + \phi(x; t)\left(\frac{\partial}{\partial x}\pi^-(x)\frac{1}{N} + \frac{\partial^2}{\partial x^2}\pi^-(x)\frac{1}{2N^2}\right) + \pi^-(x)\frac{\partial}{\partial x}\phi(x; t)\frac{1}{N} \\ &\quad + \left(\frac{\partial}{\partial x}\phi(x; t)\right)\left(\frac{\partial}{\partial x}\pi^-(x)\right)\frac{1}{N^2} + \pi^-(x)\frac{\partial^2}{\partial x^2}\phi(x; t)\frac{1}{2N^2} \\ &= \phi(x; t)\frac{\partial}{\partial x}(\pi^-(x) - \pi^+(x))\frac{1}{N} + \phi(x; t)\frac{\partial^2}{\partial x^2}(\pi^-(x) + \pi^+(x))\frac{1}{2N^2} \\ &\quad + (\pi^-(x) - \pi^+(x))\frac{\partial}{\partial x}\phi(x; t)\frac{1}{N} + (\pi^-(x) + \pi^+(x))\frac{\partial^2}{\partial x^2}\phi(x; t)\frac{1}{2N^2} \\ &\quad + \frac{\partial}{\partial x}\phi(x; t)\left(\frac{\partial}{\partial x}\pi^-(x) + \frac{\partial}{\partial x}\pi^+(x)\right)\frac{1}{N^2} \\ &= \frac{\partial\phi(x; t)(\pi^-(x) - \pi^+(x))}{\partial x}\frac{1}{N} + \frac{\partial^2\phi(x; t)(\pi^-(x) + \pi^+(x))}{\partial x^2}\frac{1}{2N^2}.\end{aligned}$$

Then, Eqs. (6) and (7) can be obtained.

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